Problem 5106. Let a, b and c be the sides of an acute angled triangle ABC. Let H be the orthocenter and let d_a, d_b and d_c be the distances from H to the sides BC, CA and AB respectively. Show that

$$d_a + d_b + d_c \leq \frac{3}{4}D$$

where D is the diameter of the circumcircle.

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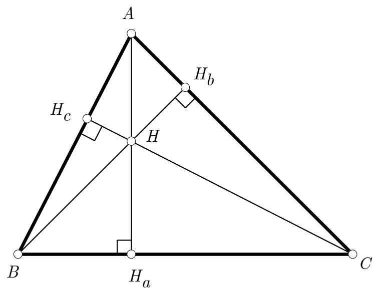

Let H_a, H_b, H_c be the feet of A, B, C onto the sides BC, CA, AB respectively and let R be the circumradius of $\triangle ABC$. We have

$$d_a = BH_a \cdot \tan(90^\circ - C) = c \cos B \cot C$$

Hence, taking into account of extended sine law, we get

$$d_a = 2R \sin C \cos B \cot C = 2R \cos B \cos C \tag{1}$$

Now, by using (1) and its cyclic permutations, the given inequality rewrites as

$$2R \cos B \cos C + 2R \cos C \cos A + 2R \cos A \cos B \leq \frac{3}{4} \cdot 2R$$

$$\cos B \cos C + \cos C \cos A + \cos A \cos B \leq \frac{3}{4} \tag{2}$$

which is true. In fact, from the well known formulas

$$\sum \cos^2 A = 1 - 2 \cos A \cos B \cos C$$

and

$$0 \leq \cos A \cos B \cos C \leq \frac{1}{8}$$

both valid for an acute-angled triangle, we get immediately

$$\sum \cos^2 A \geq \frac{3}{4} \tag{3}$$

Hence, by applying the known inequality

$$1 < \cos A + \cos B + \cos C \leq \frac{3}{2}$$

we obtain

$$\begin{aligned}
(\cos A + \cos B + \cos C)^2 &\leq \frac{9}{4} && \Rightarrow \\
\sum \cos^2 A + 2\sum \cos B \cos C &\leq \frac{9}{4} && \Rightarrow \\
2\sum \cos B \cos C &\leq \frac{9}{4} - \sum \cos^2 A \leq \frac{9}{4} - \frac{3}{4} = \frac{3}{2} && \Rightarrow \\
\sum \cos B \cos C &\leq \frac{3}{4}
\end{aligned}$$

and the conclusion follows. Equality holds for $a = b = c$. $\square$